Exercises for the course

"Mathematical modeling in biology"

Sheet 12

Exercise 48: Diffusion approximation of a nearly critical Galton-Watson process Recall the binary Galton-Watson process from Exercise 40. Let the initial number $G_0^{(n)} \in \mathbb{N}_0$ of individuals scale with $n \in \mathbb{N}$, that is, $G_0^{(n)}/n \to x \in [0,\infty)$ as $n \to \infty$. The branching rate per individual is $\beta \in (0,\infty)$. Assume that this process is nearly critical in the following sense: Let the probability $p_n \in (0,1)$ of having 2 children scale such that $n \cdot (2p_n - 1) \to s \in \mathbb{R}$ as $n \to \infty$, that is, $p_n = \frac{1}{2} + \frac{s}{2n} + o(\frac{1}{n})$ as $n \to \infty$. What is the diffusion approximation of the rescaled and accelerated process $\big(G_{tn}^{(n)}/n\big)_{t\geq 0}$ for large $n \in \mathbb{N}_0$?

Exercise 49: Itô's formula Let $(B_t)_{t\geq 0}$ be a standard Bronian motion. Define

$$X_t = c\exp(\beta B_t + \alpha t), \quad t \in [0,\infty),$$

where $c, \alpha, \beta \in (0,\infty)$. Apply Itô's formula to $f(t,x) := c\exp(\beta x + \alpha t)$, $t, x \in [0,\infty)$, and show that $X_t = f(t, B_t)$ satisfies the stochastic differential equation

$$dX_t = (\alpha + \tfrac{1}{2}\beta^2)X_t\, dt + \beta X_t dB_t$$

and $X_0 = c$.

Exercise 50: Diffusion approximation of a branching-coalescing process Recall the branching-coalescing process $(X_t^{(n)})_{t\geq 0}$ from Exercise 42 with branching rate $\beta \in (0,\infty)$ per individual, probability $p_n \in (0,1)$ of having 2 children, immigration rate $\iota_n \in (0,\infty)$ and coalescence rate $\gamma_n \in (0,\infty)$ per pair of individuals. Let the initial number $X_0^{(n)} \in \mathbb{N}_0$ of individuals scale with $n \in \mathbb{N}$, that is, $X_0^{(n)}/n \to x \in [0,\infty)$ as $n \to \infty$. Assume that the parameters scale as follows: $n(2p_n - 1) \to s \in \mathbb{R}$, $n\iota_n \to \rho \in [0,\infty)$ and $n^2\gamma_n \to c \in [0,\infty)$ as $n \to \infty$. What is the diffusion approximation of the rescaled and accelerated process $\big(X_{tn}^{(n)}/n\big)_{t\geq 0}$ for large $n \in \mathbb{N}_0$?

Exercise 51: Itô's formula and the Itô isometry Apply the Itô isometry to calculate the first two moments $\mathbb{E}^x[X_t]$, $t \in [0,\infty)$, and $\mathbb{E}^x[(X_t)^2]$, $t \in [0,\infty)$, of the following processes

- $dX_t = t dB_t$, that is, $X_t = x + \int_0^t s dB_s$,
- Feller diffusion with immigration: $dX_t = \alpha\, dt + \sqrt{\beta X_t} dB_t$

Hint: Use Itô's formula for the second moment of the second process.