Exercises for the course

"Mathematical modeling in biology"

Sheet 10

Exercise 40: Infinitesimal mean of the binary Galton-Watson process
Let $(G_t)_{t\geq 0}$ be a binary Galton-Watson process in continuous time in which each individual branches at rate $\beta \in (0,\infty)$ and leaves 2 children with probability $p \in (0,1)$ and 0 children with probability $1-p$. All involved Poisson processes are independent. Determine the infinitesimal mean and the infinitesimal variance of this process.

Exercise 41: The Brownian motion
Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For each of the following processes $(X_t)_{t\geq 0}$, show that $\mathbb{E}[X_t] = 0$ and that $\mathbb{E}[(X_t)^2 - t] = 0$ for all $t \in [0,\infty)$:

- Fix $c \in (0,\infty)$. Define $X_t := \frac{1}{c} B_{tc^2}$ for $t \in [0,\infty)$.
- Fix $s \in [0,\infty)$. Define $X_t := B_{s+t} - B_t$ for $t \in [0,\infty)$.
- Define $X_0 = 0$ and $X_t := t B_{\frac{1}{t}}$ for $t \in (0,\infty)$.

Remark: A characterization of Lèvy then essentially implies that these processes are again standard Brownian motions.

Exercise 42: Infinitesimal mean of the Branching-Coalescing process
Let $(X_t)_{t\geq 0}$ be a branching-coalescing process with immigration in which each individual branches at rate $\beta \in (0,\infty)$ and leaves 2 children with probability $p \in (0,1)$ and 0 children with probability $1-p$. Moreover, (single) individuals immigrate into the system at rate $\iota \in [0,\infty)$. Moreover, each pair of individuals "coalesces" into one individual at rate $\gamma \in (0,\infty)$. All involved Poisson processes are independent. Determine the infinitesimal mean and the infinitesimal variance of this process.

Exercise 43: Infinitesimal mean of the Brownian motion
Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Determine the infinitesimal mean and the infinitesimal variance of this process.