Exercises for the course

"Mathematical modeling in biology"

Sheet 08

Exercise 32: Critical binary Galton-Watson process II

Recall the critical binary Galton-Watson process $(G_n)_{n\in\mathbb{N}_0}$ and the moment generating function ϕ from Exercise 27. Let the process start with one individual, that is, $G_0 = 1$ almost surely. If this individual has no offspring, then the process dies out and $G_n = 0$ for all $n \geq 1$. If the first individual has two children, then the two subtrees founded by these two children are independent and the distribution of each subtree is equal to the distribution of the original process $(G_n)_{n\in\mathbb{N}_0}$, that is,

$$\mathbb{P}(G_{n+1} = k | G_1 = 2) = \mathbb{P}(G_n + \tilde{G}_n = k) \quad \text{for all } n \in \mathbb{N}_0$$

where $(G_n)_{n\in\mathbb{N}_0}$ and $(\tilde{G}_n)_{n\in\mathbb{N}_0}$ are independent and identically distributed. Using this, show that the moment generating function $\psi_n(s) = \mathbb{E}s^{G_n}$ of G_n satisfies

$$\psi_{n+1}(s) = \phi(\psi_n(s)) \qquad \text{for all } s \in [0,1], n \in \mathbb{N}_0.$$

This is a recursive equation for $n \mapsto \psi_n(s)$. Find the stable fixed point of this equation. What is the limit of $\psi_n(s)$ as $n \to \infty$ for fixed $s > 0$? Guess what this could mean for the long-time behavior of the critical binary Galton-Watson process $(G_n)_{n\in\mathbb{N}_0}$.

Exercise 33: Epidemiologisches Modell (SIR)

In der Vorlesung hatten wir die Ausbreitung einer Krankheit in einer Population deterministisch modelliert. Nun betrachten wir das stochastische Analogon. Wir führen folgende Notation und Annahmen ein:

- Sei S_t die (zufällige) Anzahl von Infizierbaren zur Zeit $t \geq 0$.
- Sei I_t die Anzahl von Infizierten zur Zeit $t \geq 0$.
- Sei R_t die Anzahl von Genesenen zur Zeit $t \geq 0$.
- Sei χ die Kontaktrate für ein gegebenes Paar von Infizierbarem und Infizierten.
- Sei τ die Wahrscheinlichkeit, dass ein Kontakt zur Infektion führt.
- Sei ρ die Genesungsrate.
- Da Genesene Antikörper gebildet haben, sind sie zunächst immun, werden aber zu Infizierbaren mit Rate σ.
- Nicht-Infizierte sterben mit Rate d.
- Infizierte sterben mit Rate δ.
- Neue Infizierbare immigrieren in das System mit Rate μ.

Was sind die Übergangsraten des Prozesses $(S_t, I_t, R_t)_{t\geq 0}$?

Exercise 34: Construction of the random walk In the lecture, we constructed the symmetric nearest-neighbour random walk on $\mathbb{Z}$ by flipping independent coins at the times of a rate 1 Poisson process and letting the random walker jump to the right at each 'heads' and to the left at each 'tails'.
Now consider the following construction: Let $\{(N_t^+(i))_{t\geq 0}, N_t^-(i) \colon i \in \mathbb{Z}\}$ be independent Poisson processes with rate $\rho \in (0,\infty)$. Let $(T_k^+(i))_{k\in\mathbb{N}}$ and $(T_k^-(i))_{k\in\mathbb{N}}$ for $i \in \mathbb{Z}$ be the respective jump times. If the random walker is in $i \in \mathbb{Z}$ and the next clock ringing at i is a $+$-clock, then the random walker jumps to the right. If the next clock ringing at i is a $-$-clock, then the random walker jumps to the left. Determine $\rho \in (0,\infty)$ such that the two constructions yield the same process (in distribution) and explain why the two constructions yield the same process (in distribution).

Exercise 35: Poisson processes There are (exactly) two telephone sets A and B in an office. Assume that the number of telephone calls at A and B are two independent Poisson processes $(N_t^A)_{t\geq 0}$ and $(N_t^B)_{t\geq 0}$ with parameters $\rho_A \in (0,\infty)$ and $\rho_B \in (0,\infty)$, respectively.

a) Let $M_t := N_t^A + N_t^B$ be the total number of telephone calls in the office up to time $t \geq 0$. Explain in words why $(M_t)_{t\geq 0}$ is again a Poisson process. Determine its parameter. Hint: Use properties of the Poisson process from the lectures.

b) Let T_1^A be the time until the first call on set A, let T_1^B be the time until the first call on set B, and let $\bar{T}_1$ be the time until the first telephone call in the office. What is the probability $\mathbb{P}(T_1^A < T_1^B) = \mathbb{P}(\bar{T}_1 = T_1^A)$ that the first call in the office is on telephone set A? What is $\mathbb{P}(T_1^A > T_1^B) = \mathbb{P}(\bar{T}_1 = T_1^B)$? Hint: You may view the calls on set A as a 'thinning' (with a certain probability to be determined) of all calls.

c) What is the expected number of calls on set A before the first call on set B? Hint: You need to determine $\mathbb{E}[N^A_{T_1^B}]$. Consider time $\bar{T}_1$: Either you have $\bar{T}_1 = T_1^A$ or you have $\bar{T}_1 = T_1^B$. How does N^A change in these two cases? Then restart the processes at time $\bar{T}_1$ and you obtain an equation for $\mathbb{E}[N^A_{T_1^B}]$ which you need to solve.