Exercises for the course

"Mathematical modeling in biology"

Sheet 07

Exercise 28: Wright-Fisher model with mutation

Recall the Wright-Fisher model from the lecture. Now we add mutation. Assume that at every birth event there is a certain probability of mutation. The mutation probability from type 'a' to 'A' is denoted as μ_{01} and the mutation probability from type 'A' to 'a' is denoted as μ_{10}. The total population size is assumed to be constantly equal to $N \in \mathbb{N}$. Let the relative frequency of type 'A' be denoted by P_n. Write down the transition probabilities of the process $(P_n)_{n\in\mathbb{N}_0}$. A picture might be helpful for this. Having done that, calculate

$$\mathbb{E}\big[P_{n+1}|P_n\big] \quad \text{and} \quad \mathrm{Var}\big[P_{n+1}|P_n\big]$$

for all $n \in \mathbb{N}_0$.

Exercise 29: Critical binary Galton-Watson process

Consider a branching process $(G_n)_{n\in\mathbb{N}_0}$ in which each mother has zero or two daughters and both possibilities occur with equal probability. This is a critical binary Galton-Watson process. More formally, let N be the (random) number of offspring. We assume that

$$\mathbb{P}(N=0) = \frac{1}{2}, \quad \mathbb{P}(N=2) = \frac{1}{2}.$$

What is the expected number μ of offspring per mother? Moreover, calculate the variance σ^2 of the offspring number. Assume $G_0 = 1$ almost surely. The lecture provides formulas for the expected number $\mathbb{E}G_n$ of indivduals in generation n and for the variance $\mathrm{Var}\, G_n$. What are these quantities in the case of critical binary branching? In addition show that the function $\phi\colon [0,1] \to [0,1]$, $s \mapsto \mathbb{E}s^N$ is a second-order polynomial in s.

Remark: The function ϕ is the so-called moment generating function. It generates the first moment as follows

$$\mathbb{E}\big[N\big] = \frac{d}{ds}\mathbb{E}\big[s^N\big]|_{s=1}.$$

Higher moments can be obtained from higher order derivatives.

Exercise 30: Sweeps

A simple model for a selective sweeps is as follows. Consider a locus on which a favorable alleles B arises at time $t=0$. The wild type is b. Let p_t be the fraction of B's at time $t \geq 0$. This fraction of B's appriximately satisfies

$$\frac{d}{dt}p_t = sp_t(1-p_t), \quad p_0 = \frac{1}{N} \tag{1}$$

for $t \geq 0$ if the population size $N \in \mathbb{N}$ is very large (so we consider a deterministic model in this exercise). We call this the logistic sweep model.

Now consider a nearby neutral locus with alleles A and a. Let v_t be the frequency of allele A in the population carrying B and let w_t be the frequency of allele A in the population carrying b at time $t \geq 0$. Recombinations between the two loci occur at rate ρ. Draw a flow diagram and write down the equations for p_t, v_t and w_t.

Exercise 31: Conditional expectation
Let X and Y be the results of throwing two independent dice. Calculate

- $\mathbb{E}(X|Y)$,
- $\mathbb{E}(X+Y|Y)$,
- $\mathbb{E}(X \cdot Y|Y)$,
- $\mathbb{P}(X+Y \in \{7,11\}|Y)$ and
- $\mathbb{E}(X|X+Y)$. Hint: $\mathbb{E}(X+Y|X+Y) = X+Y$.

Hint: You might want to use the calculation rules introduced in the lecture.