Exercises for the course

"Mathematical modeling in biology"

Sheet 05

Exercise 19: Age structured population competing for resources

Consider an age structured population with immigration. Assume that individuals of age $a \in [0, \infty)$ immigrate at rate $\mu(a)$. This rate is low for very young and very old individuals. The birth rate per individual at age a is assumed to be $\beta(a)$. The natural death rate of individuals of age a is $\delta(a)$. Moreover we assume that individuals consume resources. To model the competition for resources, we assume that the per capita death rate is equal to $\delta(a)$ plus a rate which depends on the population (no direct dependence on the resources). How could this additional death rate look like? Follow the steps of the lecture and write down a model in discrete time. Let the time interval be Δt. Letting $\Delta t \to 0$, you arrive at a differential equation involving $\frac{\partial}{\partial a} x(t, a)$ where $x(t, a)$ is the expected number of individuals at time $t \in [0, \infty)$ of age $a \in [0, \infty)$.

Exercise 20: Equilibria and stability

The population of a certain species subjected to a specific kind of predation is modelled by the difference equation

$$x_{n+1} = \rho \frac{x_n^2}{b + x_n^2} =: f(x_n) \qquad n \in \mathbb{N}_0$$

where $\rho, b > 0$. Determine the equilibria. Sketch the graph of f (mind the number of equilibria, $f(0)$ and $\lim_{x \to \infty} f(x)$). From this qualitative picture of f, can you read off the stability of each equilibrium? Then show that if $\rho^2 > 4b$, then it is possible for the population to be driven to extinction if it becomes less than a critical size. Determine the set of initial points from which the system converges to zero. The graphical technique of cobwebbing might be helpful here.

Note that the logistic growth model does not have this feature. In that model, the system converges to the carrying capacity whenever it starts from a strictly positive value.

Exercise 21: Central limit theorem

John Q. Public throws a (six-sided) dice ninety times. How many 1's do you expect? What is the distribution of the number of 1's? Which expression would you have to evaluate numerically in order to calculate the probability that John throws exactly ten 1's? Note that this expression is difficult to evaluate numerically as e.g. 90! has 138 decimals. Instead one can use the central limit theorem which you should recall from the lecture. Use the central limit theorem to obtain an approximation of the probability to see exactly ten 1's when throwing a dice ninety times.

Exercise 22: Expected values
Calculate the expected value

- of the Bernoulli distribution $\mathrm{Ber}(p)$
- of the uniform distribution on $[0, 1]$
- of the exponential distribution $\mathrm{Exp}(\lambda)$ (integration by parts might be helpful)
- of the Poisson distribution $\mathrm{Pois}(\lambda)$. Hint:

$$\frac{k}{k!} = \frac{1}{(k-1)!} \text{ and } \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = 1$$

- of the normal distribution $\mathcal{N}(\mu, \sigma^2)$. Hint: $x \mapsto x \exp\left(-\frac{x^2}{2\sigma^2}\right)$ is symmetric around 0, so

$$\int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = 0.$$

Exercise 23: Variances (Optional Exercise)
Calculate the variance $\mathrm{Var}(X) = \mathbb{E}\Big(X^2\Big) - \Big(\mathbb{E}(X)\Big)^2$

- of the Bernoulli distribution $\mathrm{Ber}(p)$
- of the uniform distribution on $[0, 1]$
- of the Poisson distribution $\mathrm{Pois}(\lambda)$. Hint: Calculate $\mathbb{E}\big(X(X-1)\big)$ and use this to obtain the variance.