Exercises for the course

"Mathematical modeling in biology"

Sheet 04

Exercise 14: Age structure
Consider an age structured population. Let $\beta(a)$ be the per capita birth rate at age $a \in [0, \infty)$ and let $\delta(a)$ be the per capita death rate at age a. We derived in the course a criterion on $\beta(\cdot)$ and $\delta(\cdot)$ when the population grows to infinity and when it decreases to zero. Use this formula to answer the following questions. Consider the following simplified situation of human beings. Assume that women only give birth between age 20 and age 50. In this time interval, the birth rate is assumed to be constantly equal to γ.

1. Suppose that the death rate of women is zero before age 60 and is constantly equal to η after that age. Under which conditions on γ and η is the population increasing and when is the population decreasing? Rephrase your answer in terms of the expected number of children. Give an interpretation why your answer is independent of η.
2. Now assume the death rate of women to be constantly equal to ε before age 60 and to be constantly equal to η after age 60. Under which conditions on γ, ε and η is the population increasing and when is it decreasing?

Exercise 16: Blower's model for HIV-progression In a Science paper from 2000, Blower et al. asked the following question. Antiretroviral therapies prevent individuals having HIV from getting AIDS. The reduces the fear of getting HIV. As consequence,

1. risky behaviour increases and
2. HIV has more time to develope resistance to drugs.

Could these effects increase the progession of HIV?

Construct a process modeling the expected numer of infected individuals in a metropolis like San Francisco. Incorporate at least the following:

- An HIV-infected individual is in an ART or not.
- ART reduces the infectivity of individuals being in an ART.
- ART increases the time until AIDS.
- In this exercise, we ignore the possibility of resistances.

Check your result intensively!

Exercise 17: Equilibria and stability Consider the recursive equation

$$x(n+1) = x(n) + \rho\, x(n) \left((x(n))^2 - \tfrac{3}{2} x(n) + \tfrac{1}{2} \right) \quad n \in \mathbb{N}_0$$

where $\rho > 0$ is a parameter. Determine the equilibria of this model. Which of these equilibria are locally unstable and which are locally stable. Determine the value ρ_0 of ρ when all equilibria become unstable. From what you have learned about pitchfork bifurcations in the lecture, guess the qualitative behavior of the model for ρ slightly greater than ρ_0.

Exercise 18: Ricker model
According to the recursion equation

$$x(n+1) = x(n) + \rho x(n) \Big(1 - \frac{x(n)}{\kappa} \Big) \qquad \forall\, n \in \mathbb{N}_0 \tag{1}$$

of the logistic growth model, it is possible for $x(n+1)$ to be negative even if $x(n)$, ρ and κ are all positive. To see this, calculate $x(1)$ by hand using $\rho = 1$, $\kappa = 100$ and starting from the population sizes $x(0) = 50$, 100 and 200. By rearranging the recursion equation, determine the population size $x(n)$ above which $x(n+1)$ becomes negative and the population goes extinct. That is, find x^* in terms of ρ and κ such that $x(n+1) < 0$ whenever $x(n) > x^*$. Check that your answer to this question is consistent with your answer to the first question of this exercise.
An alternative to the logistic growth model is the Ricker model, which has been used quite frequently in the literature. The assumption here is that the reproduction factor decreases exponentially. The recursive equation is then

$$x(n+1) = x(n)(1+\rho)e^{-\alpha x(n)} \qquad \forall\, n \in \mathbb{N}_0. \tag{2}$$

The model is known as the Ricker model. According to the recursion equation (2), is it possible for $x(n+1)$ to be negative if $x(n)$ is positive?