Exercises for the course

“Mathematical modeling in biology”

Sheet 02

Exercise 6: Haploid model of natural selection with mutation

Consider two types of individuals (type 'A' and 'a'). A type 'A' individual has an expected number β_A of offspring. A fraction μ_{Aa} hereof is of type 'a' and the remaining fraction $1-\mu_{Aa}$ is of type 'A'. A type 'a' individual has an expected number β_a of offspring. A fraction μ_{aA} hereof is of type 'A' and the remaining fraction $1-\mu_{aA}$ is of type 'a'. Let $x_A(n)$ be the expected number of type 'A' and let $x_a(n)$ be the expected number of type 'a'. Find the relation between $\big(x_A(n+1), x_a(n+1)\big)$ and $\big(x_A(n), x_a(n)\big)$. Then define

$$p(n) := \frac{x_A(n)}{x_A(n)+x_a(n)} \qquad \forall\, n \in \mathbb{N}_0. \tag{1}$$

Find the relation between $p(n+1)$ und $p(n)$. Once you have found this relation, specify constraints on $p(n)$ and on the parameters. Finally derive a continuous time version of the dynamics

Exercise 7: Pyrines and pyrimidines

The genome of any organism consists of a number of purine nucleotides (adenine(A) and guanine(G)) and pyrimidine nucleotides (cytosine(C) and thymine(T)). During DNA replication, mutations occasionally occur. A mutation can cause a purine to be copied into a pyrimidine or vice versa. Find a model in continuous time for the expected number $u(t)$ of purines and the expected number $y(t)$ of pyrimidines. First formulate suitable assumptions on the transition rates. Then draw a flow diagram and, finally, write up the model equations. There should be two parameters in your model equations. Give the interpretation of each parameter (example from the course: β is the expected number of female offspring per mother and per day).

Exercise 8: Yeast in a chemostat

Yeast cells can be grown so that it divides continually using a “chemostat”. Chemostats are tanks carrying a complete medium with all of the sugars and essential elements necessary for microbial growth. New medium is added to the tank via a constant inflow, while there is a constant outflow of liquid from the tank. Assume that the liquid in the tank is well mixed all of the time. To model the dynamics of a yeast population grown in a chemostat

1. decide for discrete or continuous time,
2. list all of the variables that you would like to include,
3. list all of the parameters that you think might be relevant,
4. specify any restrictions on the variables and parameters,
5. qualitatively describe the transitions of the variables,
6. draw a flow diagram,
7. translate the flow diagram into equations.

Finally check your result.

Exercise 9: Plankton model
Plankton consist of any drifting organisms in the open seas and is - by definition - unable to resist ocean currents. Phytoplankton consist of tiny plants which live on photosynthesis. Let $p(t)$ be the amount of phytoplankton at time t. As the amount of phytoplankton raises, fewer and fewer light reaches each particle. So the reproduction rate should decrease with the total mass. Assume that the reproduction rate of each phytoplankton particle decreases linearly with $p(t)$. Moreover, there is herbivore zooplankton which lives on phytoplankton. Let $h(t)$ be the amount of herbivore zooplankton at time t. The reproduction rate of the herbivore plankton is increasing with the amount of phytoplankton. Of course the reproduction rate should not exceed a certain upper bound. To ensure this, assume that the per capita reproduction rate of herbivore zooplankton is a multiple of $\frac{p(t)}{1+p(t)}$. For its reproduction, herbivore zooplankton consumes phytoplankton. Suppose that for reproducing one unit of herbivore zooplankton, ten times the amount of phytoplankton is needed. As a last assumption, suppose that herbivore zooplankton dies at a constant per capita rate. What you should do: Give names to all of the involved parameters. Then draw the flow diagram including all transition rates. Finally write down the model equations.

Exercise 10: Taylor's theorem Linearizing expressions often simplifies models but keeps the qualitative behavior. Sometimes higher orders are needed. Here is the general version:

Taylor's theorem:*Let $n \geq 1$ be an integer and let the function $f\colon \mathbb{R} \to \mathbb{R}$ be $n+1$ times continuously differentiable at a point $a \in \mathbb{R}$. Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(y)}{n!}(x-y)^n \, dy \tag{2}$$

for x in a neighborhood of a where $R_n(x) := \int_a^x \frac{f^{(n+1)}(y)}{n!}(x-y)^n \, dy$ satisfies that

$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0. \tag{3}$$

that is, $R_n(x) = o(|x-a|^n)$ as $x \to a$.

Consequently, a smooth function can be approximated by the so-called n-th Taylor polynomial around a

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{4}$$

If this converges as $n \to \infty$, then we get the Taylor series around a

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{5}$$

In order to calculate the Taylor series around a, you need to calculate the derivatives of the function at hand at the point a.

Fix $m \in \mathbb{N}$. Calculate the Taylor series of

- e^x around $a = 0$
- $\log(x)$ around $a = 1$
- $(1+x)^m$ around $a = 0$
- $\frac{1-x^{m+1}}{1-x}$ around $a = 0$

Most of the time, we are interested in linearizations. Write down $P_1(x)$ of these functions. Moreover write down $P_1(x)$ of the following functions

- $\sin(x)$ around $a = 0$
- $\cos(x)$ around $a = 0$
- $(e^x + x)^m$ around $a = 0$
- $\frac{e^x - 1}{x}$ around $a = 0$

Example: The Taylor series of $(1+x)^2$ around 0 is $1 + 2x + x^2$ and $P_1(x)$ is $1 + 2x$.