Exercises for the course

"Mathematical modeling in biology"

Sheet 01

The first exercise sheet contains rather simple exercises to get familiar with difference equations and with differential equations.

Exercise 1: Bank account of a consumer with income
Consider the saving account of John Q. Public. John has €1000 on his account at the end of this year. His yearly income is €20.000 after tax. Adding up all of his expenses, the annual costs are €15.000. John saves the rest of his income on his saving account. At the end of each December, the bank adds 3% interest to the account. Assume that the savings of one year already yield interest in that year. Model the saving account of John by writing down an equation for the amount $x(n)$ of Euro on John's account. If you wish to you could figure out the amount on John's account at the end of 2020.

Exercise 2: Model for the expected number of female mice on Garfield's farm
We wish to model the expected number of female mice on a farm. Time is measured in days. The model should contain the following dynamics:

- Mice have offspring
- Mice die from natural deaths
- There are no natural predators on the farm except for one cat called Garfield. Garfield tries to catch as many mice as possible. His success increases linearly with the number of mice on the farm.

Figure out a suitable model for the expected number $x(n)$ of female mice after day n. Start with suitable assumptions on the parameters (e.g. the expected number of daughters per mouse is β per day). Then decide for an ordering of the events 'birth', 'natural death' and 'deaths from predation'. Finally write down the relation between $x(n+1)$ and $x(n)$ for $n \in \mathbb{N}_0$.

(*Additional info (wikipedia): The house mouse (mus musculus) has an expectation of life (without predation) of two years on average. So it is reasonable to set the expected number of natural deaths to* $\frac{2}{365}$ *per day. Breeding occurs throughout the year. Each female can have some* $5-10$ *litters per year. Each litter consists of* $3-11$ *young with an average of* $6-8$*. So the expected number of births of daughters is roughly* $\frac{7.5 \cdot 7/2}{356}$ *per day.*)

Exercise 3: Migration in continuous time
Aiming at a simple model of migration between K locations, suppose that individuals migrate between the demes $\{1, 2, 3, \ldots, K\}$ and nothing else changes the number of individuals. Migration is assumed to happen potentially at any time. So we choose time to be continuous. Let $x_i(t)$ be the expected number of individuals on deme $i \in \{1, \ldots, K\}$ at time $t \geq 0$. Draw a flow diagram for the special case $K = 3$ and add all flow rates. It might be useful to index the migration rates as in a matrix. Then derive the differential equations for $x_1(t), \ldots, x_K(t)$.

Exercise 4: Separation of variables

The solution of the exponential growth model

$$\frac{dx(t)}{dt} = \rho\, x(t) \quad \text{for all } t \geq 0 \tag{1}$$

is given by $x(t) = x(0)\exp(\rho t)$ for $t \in [0, \infty)$. Derive this solution by using separation of variables.

The continuous time logistic growth model is given by

$$\frac{dx(t)}{dt} = \rho\, x(t)\Big(1 - \frac{x(t)}{\kappa}\Big) \quad \text{for all } t \geq 0. \tag{2}$$

The parameter $\rho \in (0, \infty)$ is the growth rate and $\kappa \in (0, \infty)$ is the carrying capacity. Find the general solution by using 'separation of variables'.

Hint: It might be useful to find constants a, b such that

$$\frac{1}{\rho y(1 - \frac{y}{\kappa})} = \frac{a}{y} + \frac{b}{1 - \frac{y}{\kappa}} \quad \textit{for all } y \in (0, \kappa). \tag{3}$$

Check your result as follows: If $x(0) = \kappa$, then $x(t) = \kappa$ should hold for all $t \geq 0$. Furthermore check $\lim_{s\to\infty} x(s) = \kappa$, $\lim_{\kappa\to\infty} x(t) = e^{\rho t}x(0)$ *and* $\lim_{\rho\to 0} x(t) = x(0)$ *for every $t \geq 0$.*

Exercise 5: Predator-prey model with two prey species

Let '0', '1' and '2' be three species of which '0' and '1' are the prey for the predator '2'. Species '0' and '1' reproduce at a constant per capita rate. The hunting success of each predator is assumed to be proportional to the expected number of prey. Assume that each unit of successfully hunted prey i is converted into ε_i units of the predator, $i = 0, 1$. Moreover the predators die at per capita rate δ. Suppose that the two prey-species do not interact. Draw the flow diagram of this two-prey, single-predator Lotka-Volterra model. Then write down the model equations.