Handout on **Quantitative Genetics** (Lecture "Evolutionary Ecology")

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Contents

Quantitative Traits

continuous traits: weight, size, growth rate. . .

discrete traits: number of offspring, bristle number,. . .

traits with quantitative thresholds: environment and genes determine whether a character is expressed

Quantitative Genetics

- natural selection needs phenotypic variation to operate
- many traits are influenced by few major and many minor genes
- Q.G. has been successfully applied in animal an plant breeding
- application to evolutionary and ecological processes not trivial
- no exact knowledge of genetic mechanisms, rather statistical approach
- QTL analysis to search for genomic regions that influence a trait

Recommended Books

References

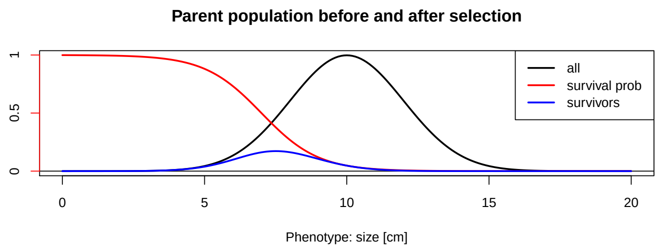
[LW98] M. Lynch, B. Walsh (1998) *Genetics and Analysis of Quantitative Traits* Sinauer Associates, Inc., Sunderland, MA, USA

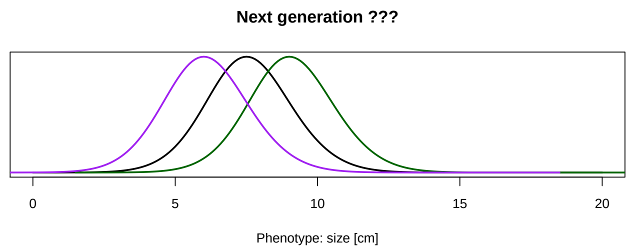
[BB+07] N.H. Barton, D.E.G. Briggs, J.A. Eisen, D.B. Goldstein, N.H. Patel (2007) *Evolution* Cold Spring Harbor Laboratory Press, Cold Spring Harbor, NY, USA

Parts of this handout are based on the book by Lynch and Walch and on material provided by Wilfried Gabriel.

1 Selection on a quantitative trait

Selection on quantitative trait

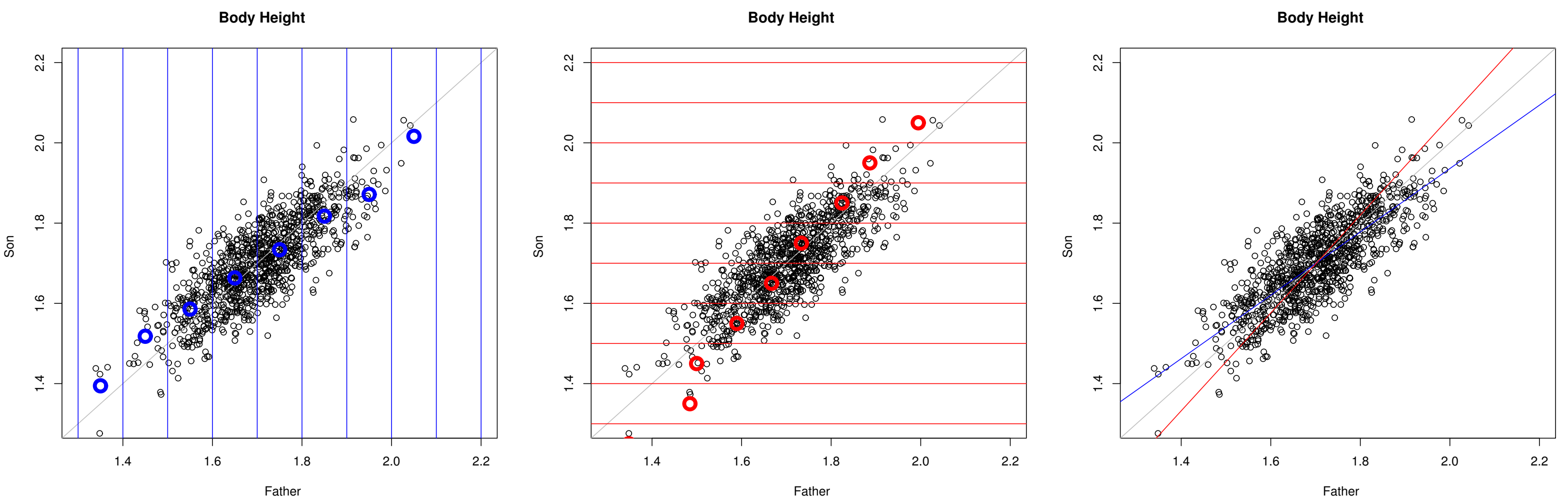




Origin of the word "Regression"

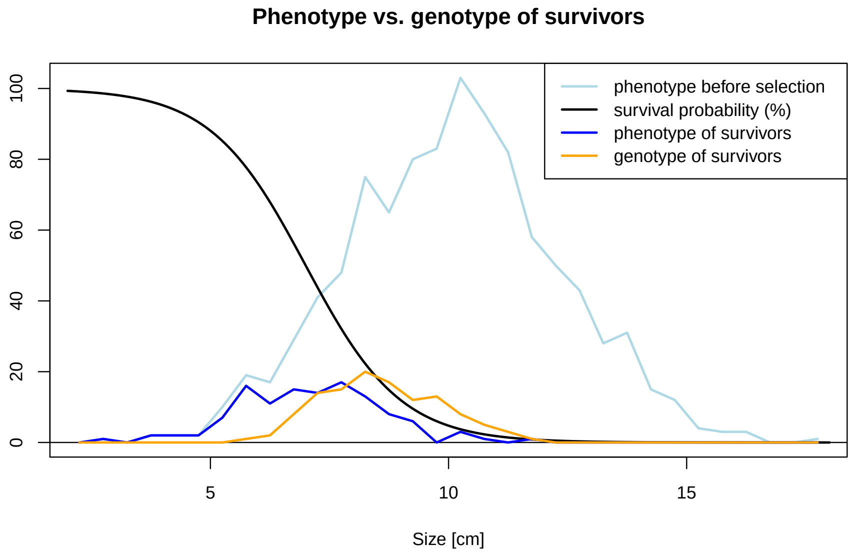
Sir Francis Galton (1822–1911): Regression toward the mean.

Tall fathers tend to have sons that are slightly smaller than the fathers.
Sons of small fathers are on average larger than their fathers.

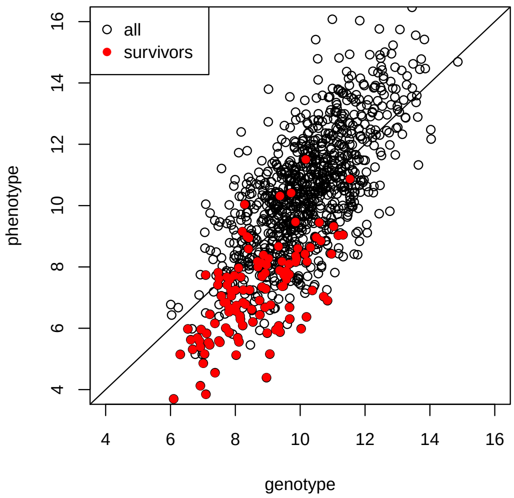


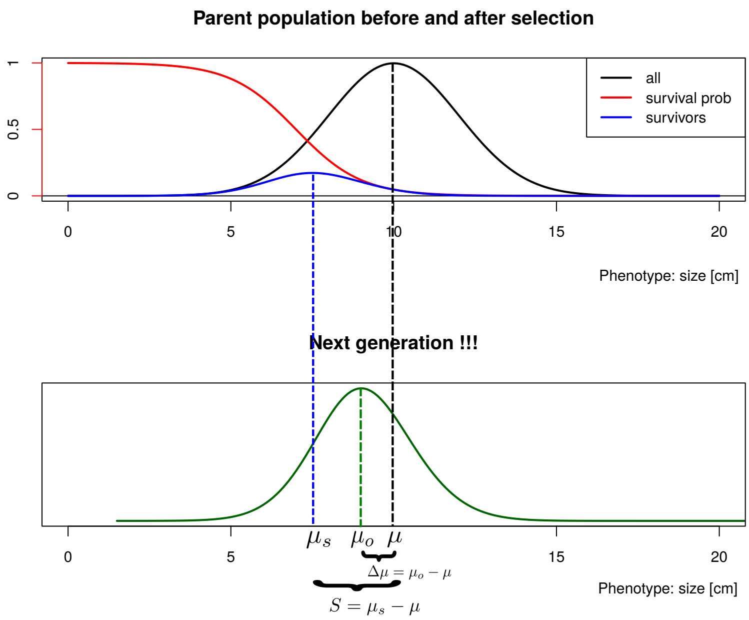
Similar effects

- In sports: The champion of the season will tend to fail the high expectations in the next year.
- In school: If the worst 10% of the students get extra lessons and are not the worst 10% in the next year, then this does not proof that the extra lessons are useful.




```
genotype <- rnorm(1000,10,1.5)
environment <- rnorm(1000,0,1.5)
phenotype <- genotype + environment

hist(phenotype,col="lightblue",breaks=4:36/2)

survival.prob <- function(x) {
  1-1/(1+exp(-x+7))
}

lines(20:180/10,survival.prob(20:180/10)*100,lwd=2)

survivors <- rbinom(1000,size=1,prob=(survival.prob(phenotype)))

hist(phenotype[survivors==1],add=TRUE,col="blue",breaks=4:36/2)
hist(genotype[survivors==1],add=TRUE,col="orange",breaks=4:36/2)
```





2 Some tools from probability theory

Definition 1 (Expectation value) *Let X be a random variable with finite or countable state space $\mathcal{S} = \{x_1, x_2, x_3 \ldots\} \subseteq \mathbb{R}$. The expectation value of X is defined by*

$$\mathbb{E}X = \sum_{x \in \mathcal{S}} x \cdot \Pr(X = x)$$

It is also common to write μ_X instead of $\mathbb{E}X$.

If we replace probabilities by relative frequencies in this definition, we get the formula for the mean value (of a sample).

Definition 2 (Expectation value) *If X is a random variable with finite or countable state space $\mathcal{S} = \{x_1, x_2, x_3 \ldots\} \subseteq \mathbb{R}$, the expectation value of X is defined by*

$$\mathbb{E}X = \sum_{x \in \mathcal{S}} x \cdot \Pr(X = x)$$

Examples:

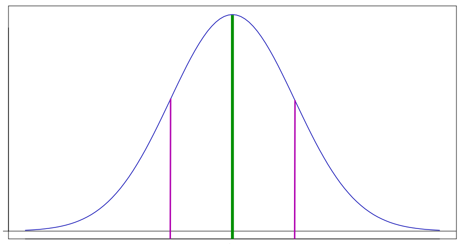
- Let X be Bernoulli distributed with success probability $p \in [0,1]$. Then we get
$$\mathbb{E}X = 1 \cdot \Pr(X=1) + 0 \cdot \Pr(X=0) = \Pr(X=1) = p$$

- Let W be the result of rolling a die. Then we get
$$\begin{aligned}\mathbb{E}W &= 1 \cdot \Pr(W=1) + 2 \cdot \Pr(W=2) + \ldots + 6 \cdot \Pr(W=6) \\ &= 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + \ldots + 6 \cdot \tfrac{1}{6} = 21\tfrac{1}{6} = 3.5\end{aligned}$$

Definition 3 (Expectation value) *If X is a random variable with continuous state space $U \subseteq \mathbb{R}$ and probability density function f the expectation value of X is defined by*

$$\mathbb{E}X = \int_U x \cdot f(x)\; dx$$

Example: Let $Z \sim \mathcal{N}(\mu, \sigma^2)$, i.e. normally distributed with density
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{s\sigma^2}},$$
then $\mathbb{E}(Z) = \mu$.



Calculating with expectatins

Theorem 1 (Linearity of Expectation) *If X and Y are random variables with values in $\mathbb{R}$ and if $a \in \mathbb{R}$, we get:*

- $\mathbb{E}(a \cdot X) = a \cdot \mathbb{E}X$
- $\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$

Theorem 2 (Only if independent!) *If X and Y are **stochastically independent** random variables with values in $\mathbb{R}$, we get*

- $\mathbb{E}(X \cdot Y) = \mathbb{E}X \cdot \mathbb{E}Y$.

But in general $\mathbb{E}(X \cdot Y) \neq \mathbb{E}X \cdot \mathbb{E}Y$. Example:

$$\mathbb{E}(W \cdot W) = \tfrac{91}{6} = 15.167 > 12.25 = 3.5 \cdot 3.5 = \mathbb{E}W \cdot \mathbb{E}W$$

Proof of Linearity: If $\mathcal{S}$ is the state space of X and Y, and if $a, b \in \mathbb{R}$, we obtain:

$$\begin{aligned}&\mathbb{E}(a \cdot X + b \cdot Y) \\ &= \sum_{x \in \mathcal{S}} \sum_{y \in \mathcal{S}} (a \cdot x + b \cdot y) \Pr(X=x, Y=y) \\ &= a \cdot \sum_{x \in \mathcal{S}} \sum_{y \in \mathcal{S}} x \Pr(X=x, Y=y) + b \cdot \sum_{x \in \mathcal{S}} \sum_{y \in \mathcal{S}} y \Pr(X=x, Y=y) \\ &= a \cdot \sum_{x \in \mathcal{S}} x \sum_{y \in \mathcal{S}} \Pr(X=x, Y=y) + b \cdot \sum_{y \in \mathcal{S}} y \sum_{x \in \mathcal{S}} \Pr(X=x, Y=y) \\ &= a \cdot \sum_{x \in \mathcal{S}} x \Pr(X=x) + b \cdot \sum_{y \in \mathcal{S}} y \Pr(Y=y) \\ &= a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y)\end{aligned}$$

Proof of the product formula: Let $\mathcal{S}$ be the state space of X and Y, and let X and Y be (stochastically) independent.

$$\begin{aligned}
&\mathbb{E}(X \cdot Y) \\
&= \sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} (x \cdot y) \Pr(X = x, Y = y) \\
&= \sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} (x \cdot y) \Pr(X = x) \Pr(Y = y) \\
&= \sum_{x\in\mathcal{S}} x \Pr(X = x) \cdot \sum_{y\in\mathcal{S}} y \Pr(Y = y) \\
&= \mathbb{E}X \cdot \mathbb{E}Y \cdot
\end{aligned}$$

Theorem 3 *If X is random variable with finite state space $\mathcal{S} \subset \mathbb{R}$, and if $f : \mathbb{R} \to \mathbb{R}$ is a function, we obtain*

$$\mathbb{E}(f(X)) = \sum_{x\in\mathcal{S}} f(x) \cdot \Pr(X = x)$$

Exercise: proof this.

Expectation of the binomial distribution

Let $Y_1, Y_2, \ldots, Y_n$ be the indicator variables of the n independent trials, that is

$$Y_i = \begin{cases} 1 & \text{if trial } i \text{ succeeds} \\ 0 & \text{if trial } i \text{ fails} \end{cases}$$

Then $X = Y_1 + \cdots + Y_n$ is binomially distributed with parameters (n, p), where p is the success probability of the trials.

Linearity of expectation implies

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}(Y_1 + \cdots + Y_n) \\
&= \mathbb{E}Y_1 + \cdots + \mathbb{E}Y_n \\
&= p + \cdots + p \;=\; np
\end{aligned}$$

Note:

$$X \sim \text{bin}(n, p) \Rightarrow \mathbb{E}X = n \cdot p$$

Definition 4 (Variance, Covariance and Correlation) *The Variance of a $\mathbb{R}$-valued random variable X is*

$$\mathit{Var}X = \sigma_X^2 = \mathbb{E}\left[(X - \mathbb{E}X)^2\right].$$

$\sigma_X = \sqrt{\mathit{Var}\ X}$ is the Standard Deviation.
If Y is enother $\mathbb{R}$-valued random variable,

$$\mathit{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}X) \cdot (Y - \mathbb{E}Y)\right]$$

is the Covariance of X and Y.
The Correlation of X and Y is

$$\mathit{Cor}(X, Y) = \frac{\mathit{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}.$$

The Variance

$$\text{Var}X = \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$$

is the average squared deviation from the expectation.

The Correlation

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

is always between in the range from -1 to 1. The random variables X and Y are

- positively correlated, if X and Y tend to be both above average or both below average.
- negatively correlated, if X and Y tend to deviate from their expectation values in opposite ways.

If X and Y are independent, they are also uncorrelated, that is $\text{Cor}(X, Y) = 0$.

Example: rolling dice

Variance of result from rolling a die W:

$$
\begin{aligned}
\text{Var}(W) &= \mathbb{E}\big[\big(W - \mathbb{E}W\big)^2\big] \\
&= \mathbb{E}\big[\big(W - 3.5\big)^2\big] \\
&= (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + \ldots + (6-3.5)^2 \cdot \frac{1}{6} \\
&= \frac{17.5}{6} \\
&\approx 2.92
\end{aligned}
$$

Example: Empirical Distribution

If $x_1, \ldots, x_n \in \mathbb{R}$ are data and if X is the result of a random draw from the data (such that $\Pr(X = x_i) = \frac{1}{n}$), we get:

$$\mathbb{E}X = \sum_{i=1}^{n} x_i \Pr(X = x_i) = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{x}$$

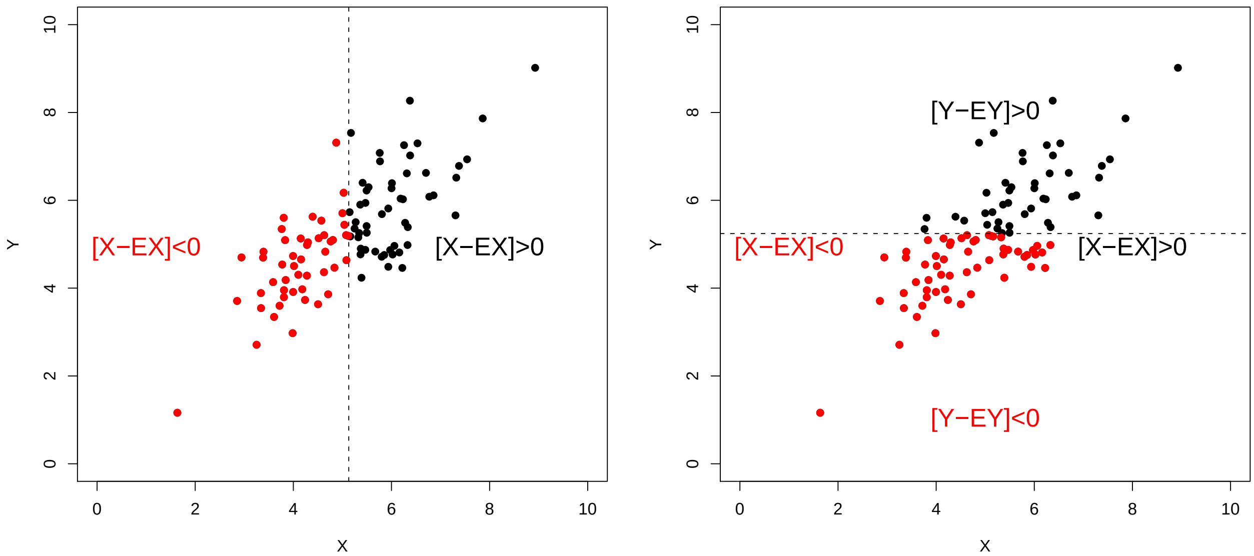
and

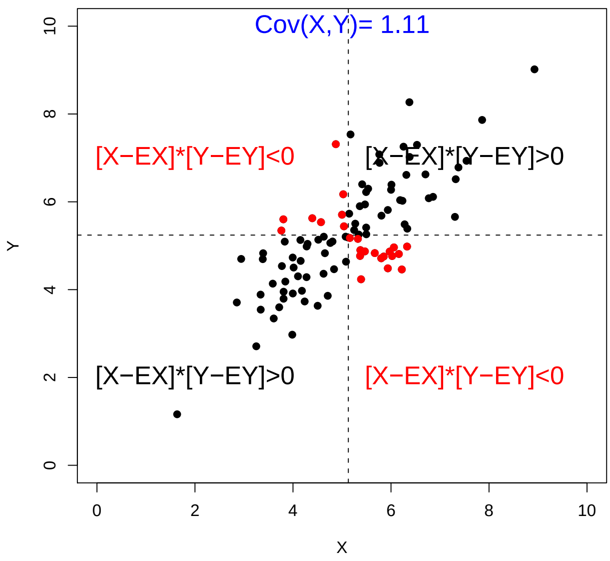
$$\text{Var } X = \mathbb{E}\big[\big(X - \mathbb{E}X\big)^2\big] = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

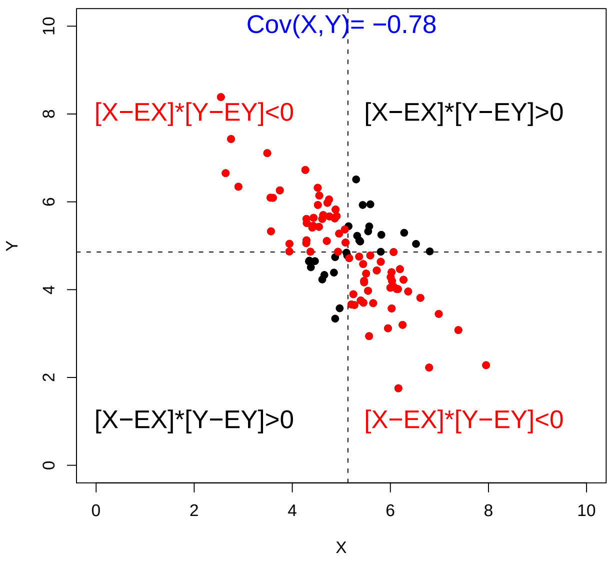
If $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R} \times \mathbb{R}$ are data if (X, Y) are drawn from the data such that $\Pr((X, Y) = (x_i, y_i)) = \frac{1}{n}$, we get

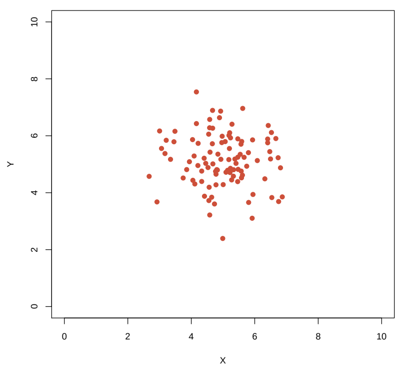
$$\text{Cov } (X, Y) = \mathbb{E}\big[\big(X - \mathbb{E}X\big)\big(Y - \mathbb{E}Y\big)\big] = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

Why Cov$(X, Y) = \mathbb{E}([X - \mathbb{E}X][Y - \mathbb{E}Y])$?

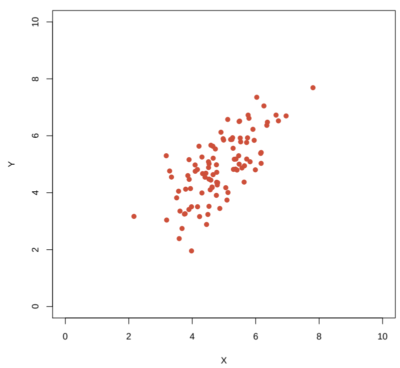




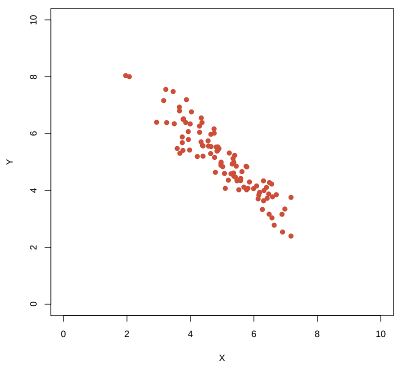




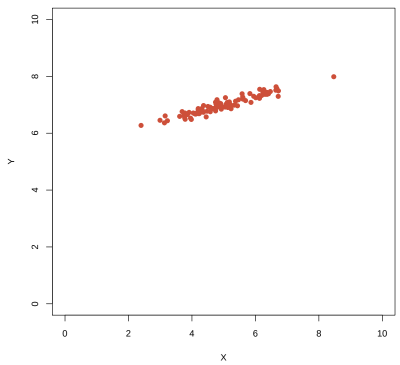
$\sigma_X = 0.95$, $\sigma_Y = 0.92$
$\mathrm{Cov}(X, Y) = -0.06$
$\mathrm{Cor}(X, Y) = -0.069$



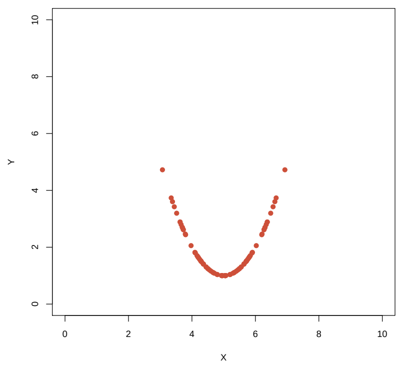
$\sigma_X = 1.14$, $\sigma_Y = 0.78$
$\mathrm{Cov}(X, Y) = 0.78$
$\mathrm{Cor}(X, Y) = 0.71$



$\sigma_X = 1.13$, $\sigma_Y = 1.2$
$\mathrm{Cov}(X,Y) = -1.26$
$\mathrm{Cor}(X,Y) = -0.92$



$\sigma_X = 1.03$, $\sigma_Y = 0.32$
$\mathrm{Cov}(X,Y) = 0.32$
$\mathrm{Cor}(X,Y) = 0.95$



$\sigma_X = 0.91$, $\sigma_Y = 0.88$
$\mathrm{Cov}(X,Y) = 0$
$\mathrm{Cor}(X,Y) = 0$

Calculation rules for variances

$\mathrm{Var}X = \mathbb{E}[(X - \mathbb{E}X)^2]$

- $\mathrm{Var}X = \mathrm{Cov}(X,X)$
- $\mathrm{Var}X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$
- $\mathrm{Var}(a \cdot X) = a^2 \cdot \mathrm{Var}X$
- $\mathrm{Var}(X+Y) = \mathrm{Var}X + \mathrm{Var}Y + 2 \cdot \mathrm{Cov}(X,Y)$
- $\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2 \cdot \sum_{j=1}^n \sum_{i=1}^{j-1} \mathrm{Cov}(X_i, X_j)$
- If (X,Y) stochastically independent we get:

$$\mathrm{Var}(X+Y) = \mathrm{Var}X + \mathrm{Var}Y$$

Calculation rules for Covariances

$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - EX) \cdot (Y - \mathbb{E}Y)]$

- If X and Y are independent, then $\mathrm{Cov}(X,Y) = 0$ (but not the other way around!)

- $\mathrm{Cov}(X,Y)$=$\mathrm{Cov}(Y,X)$
- $\mathrm{Cov}(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}X \cdot \mathbb{E}Y$
- $\mathrm{Cov}(a \cdot X, Y) = a \cdot \mathrm{Cov}(X,Y) = \mathrm{Cov}(X, a \cdot Y)$
- $\mathrm{Cov}(X + Z, Y) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(Z,Y)$
- $\mathrm{Cov}(X, Z + Y) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(X,Y)$

The last three rules describe the bilinearity of covariance.

Calculation rules for Correlations
$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$

- $-1 \leq \mathrm{Cor}(X,Y) \leq 1$
- $\mathrm{Cor}(X,Y) = \mathrm{Cor}(Y,X)$
- $\mathrm{Cor}(X,Y) = \mathrm{Cov}(X/\sigma_X, Y/\sigma_Y)$
- $\mathrm{Cor}(X,Y) = 1$ if and only if Y is an increasing, affine-linear function of X, that is, if $Y = a \cdot X + b$ for appropriate $a > 0$ and $b \in \mathbb{R}$.
- $\mathrm{Cor}(X,Y) = -1$ if and only if Y is an decreasing, affine-linear function of X, that is, if $Y = a \cdot X + b$ for appropriate $a < 0$ and $b \in \mathbb{R}$.

Note that if

$$Y = m \cdot X + b + \varepsilon,$$

where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$, that is, m is the slope of the regression line for predicting Y from X, then

$$m = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} = \mathrm{Cor}(X,Y) \cdot \frac{\sigma_Y}{\sigma_X}$$

3 Quantiative-Genetic Theory

μ mean phenotype before selection

μ_s mean phenotype after selection but before reproduction

$S = \mu_s - \mu$ **directional selection differential**

μ_o mean phenotype in offspring generation

$\Delta\mu = \mu_o - \mu$

$W(z)$ **individual fitness**: probability that individual with phenotype z will survive to reproduce

$p(z)$ density of phenotype z before selection

$\overline{W} = \int W(z) \cdot p(z)\ dz$ mean individual fitness

$w(z) = W(z)/\overline{W}$ relative individual fitness

$p_s(z) = w(z)p(z)$ density of phenotype z after selection but before reproduction (density in a stochastic sense, i.e. integrates to 1)

Let Z be the phenotype of an individual drawn randomly from the parent population before selection. Then,

$$\mu = \mathbb{E}Z \quad \text{and} \quad \mathbb{E}(w(Z)) = 1.$$

$$\Rightarrow S = \mu_s - \mu = \mathbb{E}(Z \cdot w(Z)) - \mathbb{E}(Z) \cdot \mathbb{E}(w(Z)) = \mathrm{Cov}(Z, w(Z))$$

Thus, we obtain:

Theorem 4 (Robertson-Price identity; Robertson 1966; Price 1970/72)

$$S = \mathit{Cov}(Z, w(Z))$$

Assume we can partition the phenotypic value Z into a genotypic value G and an environmental (or random) deviation E:

$$Z = G + E$$

Then,

$$\mathrm{Cov}(Z, G) = \mathrm{Cov}(G + E, G) = \mathrm{Var}(G) + \mathrm{Cov}(E, G)$$

and

$$\mathrm{Cor}(Z, G) = \frac{\mathrm{Var}(G) + \mathrm{Cov}(G, E)}{\sigma_G \cdot \sigma_Z}.$$

In the special case of $\mathrm{Cov}(G, E) = 0$, we obtain for the genetic contribution of the phenotypic variance

$$\mathrm{Cor}^2(G, Z) = \frac{\mathrm{Var}(G)}{\mathrm{Var}(Z)}.$$

(Note that if $\mathrm{Cov}(G, E) = 0$, then $\mathrm{Var}(Z)$=$\mathrm{Var}(G)$+$\mathrm{Var}(E)$)

Note that if E is really due to environmental effects, $\mathrm{Cov}(G, E)$ may not be 0 if the population is genetically and spatially structured (and for many other possible reasons).

In any case,

$$\frac{\mathrm{Var}(G)}{\mathrm{Var}(Z)} =: H^2$$

is called **heritability in the broad sense**.

Let Z_m, Z_f, Z_o be the phenotypes of a mother, a father, and their offspring, and let G_m and G_f be the phenotypic effects of the genes transmitted by the mother and the father to the offspring

Under mating is so random in the population that we can neglect many correlations (between parental genotypes and environmental effects etc.), we obtain:

$$\mathrm{Cov}\left(\frac{Z_m + Z_f}{2}, Z_o\right) = \mathrm{Cov}\left(\frac{G_m + G_f}{2}, G_m + G_f\right) = \frac{\mathrm{Var}G_m + \mathrm{Var}G_f}{2} = \frac{\sigma_A^2}{2},$$

where σ_A^2 is the additive genetic variance.

The slope of the regression line to predict offspring phenotype from midparent phenotype

$$h^2 := \sigma_A^2 / \mathrm{Var}(Z)$$

is called **narrow-sense heritability**.

Note that the narrow-sense heritability can (in contrast to the broad-sense heritability) be estimated from statistics that are easy to observe ($\mathrm{Cov}(Z_m + Zf, Z_0)$ and $\mathrm{Var}(Z)$).

Theorem 5 (breeders' equation)

$$\Delta\mu = h^2 S$$

4 Phenotypic response on environment

Genetic and environmental effects on phenotype can interact!

adaptive phenotypic plasticity: ability of a genotype to express different phenotypes in response to variation of the environment so that phenotypic changes have an adaptive value.

e.g.: **acclimation:** reversible physiological response to enhance environmental tolerance (e.g. hibernating)

constitutive defense
Erinaceus europaeus Spines of hedgehogs are not induced

inducible defense
Acacia xanthophloea

Giraffa camelopardalis Some plants let their spines grow larger if large animals have been feeding on them

Inducible Defense in Rotatoria (De Beauchamp 1952)
Brachyonus calyciflorus
gets spines if predator is present

Asplanchna

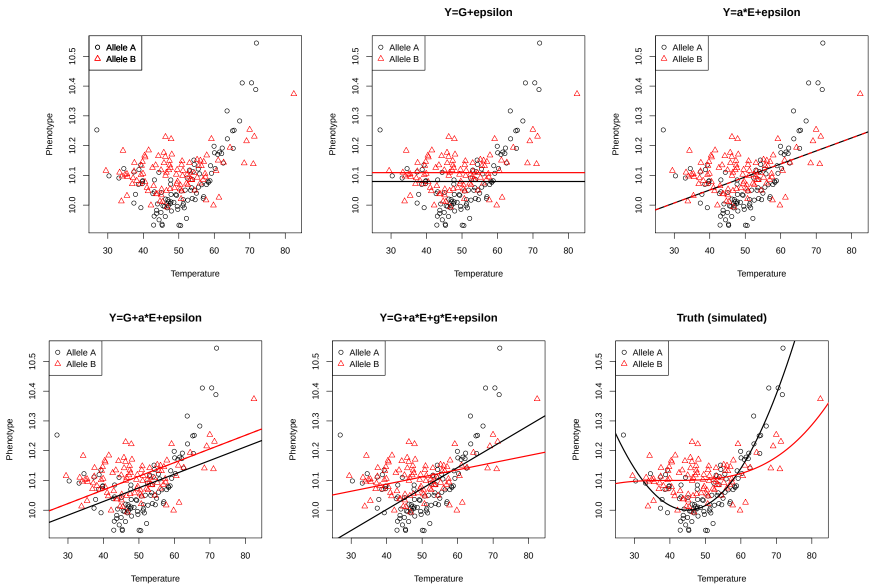
Hyla arborea
offspring of the same mother treated without (small) and with (large) the kairomon of a predator

Inducible defense in Daphnia (work of Christian Laforsch)
Life history changes: growth pattern and investment into reproduction; stay small as defense against large predators, grow large as defense against small predators

Change in behavior: dive deeper during the day if fish present

Inducible defense in *Daphnia longicephala*
can also grow larger if predator is present

Inducible reversible defense in *Daphnia cucullata*
can change the shape of their head as defense against a predator



- In models with interactions of genotypic and environmental effects, the slope of the regression line describing how the phenotype depends on the environmental parameter can depend on the genotype.
- Concepts like narrow-sense heritability or the decomposition of phentypic variance into genetic and environmental contributions is only possible in models without these interactions.
- G.E.P. Box: "Essentially, all models are wrong, but some are useful".
- The most useful models are usually not the most realistic ones.